TIME VALUE OF MONEY

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INTRODUCTION AND FUTURE VALUE

- *Interest* is paid by borrowers to lenders for the use of lenders' money. The level of interest charged is typically stated as a percentage of the *principal* (the amount of the loan).
- When a loan matures, the principal must be repaid along with any unpaid accumulated interest.

SIMPLE INTEREST

- Interest is computed on a *simple* basis if it is paid only on the principal of the loan.

- if a sum of money (X0) were borrowed at an annual interest rate i and repaid at the end of n years with accumulated interest accruing on a simple basis, the total sum repaid (FVn or Future Value at the end of year n) is determined as follows: $FV_n = X_0(1 + n_i)$.

- *Compound* interest is paid on accumulated loan interest as well as on the principal.

  <u>Simple interest:</u>

  If a consumer borrowed \$1,000 at an interest rate of 10% for one year, his total repayment would be \$1,100 :

  $FV1 = \$1,000(1 + 1 \cdot 0.1) = \$1,000 \cdot 1.1 = \$1,100.$

  If the loan were to be repaid in two years, its future value would be determined as follows:

  $FV2 = \$1,000(1 + 2 \cdot 0.1) = \$1,000 \cdot 1.2 = \$1,200$

COMPOUND INTEREST

- Interest is computed on a compound basis when the borrower pays interest on accumulated interest as well as on the loan principal.

  $FV_n = X_0(1 + i)^n.$

- An individual were to deposit $1,000 into a savings account paying annually compounded interest at a rate of 10% (the bank is borrowing money), the future value of the account after five years would be:

  $FV_5 = \$1{,}000(1 + 0.1)^5 = \$1{,}610.51$

FRACTIONAL PERIOD COMPOUNDING OF INTEREST

- In many savings accounts and other investments, interest can be compounded semiannually, quarterly, or even daily.
  $FV_n = X_0(1 + i/m)^{mn}$,
  where interest is compounded m times per year.
- if m is 2, then interest is compounded on a semiannual basis. The semiannual interest rate is simply i/m or $i/2$.
- If \$1,000 were deposited into a savings account paying interest at an annual rate of 10% compounded semiannually, its future value after five years would be \$1,628.90, determined as follows:

  $FV_5 = \$1,000(1 + 0.1/2)^{2 \cdot 5}$

Annual Percentage Yield - APY

- The annual percentage yield (APY) is the effective annual rate of return taking into account the effect of compounding interest. APY is calculated by:

$$FV = X_0\left(1+\frac{i}{m}\right)^{mn} = X_0(1+APY)^n.$$

Thus, we can compute APY as follows:

$$APY = \left(1+\frac{i}{m}\right)^{m} - 1.$$

m: periodic
i/m: periodic rate

Annual Percentage Yield - APY

- Suppose you are considering whether to invest in a one-year zero-coupon bond that pays 6% upon maturity or a high-yield money market account that pays 0.5% per month with monthly compounding.

- At first glance, the yields appear equal because 12 months multiplied by 0.5% equals 6%. However, when the effects of compounding are included by calculating the APY, the second investment actually yields 6.17%, as $(1 + .005)^{12} - 1 = 0.0617$.

APPLICATION 4.1: *APY* AND BANK ACCOUNT COMPARISONS

- Consider an example where a savings account at bank X pays 6% interest compounded daily and a similar account at bank Y pays 6.25 % interest, compounded semiannually. Which account will pay more to an investor who leaves a $100 deposit for one year (365 days)?

Solution:

- FV of X =$ 106.1831
- FV of Y = $ 106.34766
- APY of X =v0.061831311
- APY of Y= 0.063476563

CONTINUOUS COMPOUNDING OF INTEREST

- If interest were to be compounded **an infinite** number of times per period, we would say that interest is compounded continuously.

- $FV_n = X_0(1 + i/m)^{mn}$, we saw that increases in **m** cause the future value of an investment to increase. As **m** approaches infinity, **FVn** continues to increase, however at decreasing rates.
- $FV_n = X_0e^{in}$, where e is the natural log whose value can be approximated at 2.718,
- If an investor were to deposit $1,000 into an account paying interest at a rate of
  10%, continuously compounded (or compounded an infinite number of times per
  year), the account's future value would be approximately $1,648.64:
  *FV*5 = $1,000 · e1·5 ≈ $1,000 · 2.7180.5 = $1,648.64.

ANNUITY FUTURE VALUES

- An *annuity* is a series of equal payments made at equal intervals

$FVA = X[(1 + i)^{n-1} + (1 + i)^{n-2} + \ldots + (1 + i)^2 + (1 + i)^1 + 1]$
(4.9)

Multiply both of its sides by $1 + i$:

$FVA(1 + i) = X[(1 + i)^n + (1 + i)^{n-1} + \ldots + (1 + i)^3 + (1 + i)^2 + (1 + i)]$
(4.10)

Subtract equation (4.9) from equation (4.10), to obtain:

$FVA(1 + i) - FVA = X[(1 + i)^n - 1]$
(4.11)

$FVA \cdot 1 + FVA \cdot i - FVA = X[(1 + i)^n - 1] = FVA \cdot i = X[(1 + i)^n - 1]$
(4.12)

- (4.13) cash flows are paid at the end of each period $FVA = \frac{X[(1+i)^n - 1]}{i}$.

ANNUITY FUTURE VALU

- Consider an example applicable to many individuals who open *Individual Retirement Accounts* (I.R.A.'s), from which they may withdraw when they reach the age of 59 years. Consider an individual who makes a $2,000 contribution to his I.R.A. at the end of each year for 20 years. All of his contributions receive a 10% annual rate of interest, compounded annually. What will be the total value of this account, including accumulated interest, at the end of the 20-year period?

ANNUITY FUTURE VALUE

- Note that each of the above calculations assumes that cash flows are paid at the end of each period.
- If, instead, cash flows were realized at the beginning of each period, the annuity would be *referred* to as an *annuity due*. The annuity due would generate an extra year of interest on each cash flow. Hence, the future value of an annuity due is determined by simply multiplying the future value annuity formula by (1 + i):

$$FVA_{n,\mathrm{due}} = X\frac{(1+i)^n - 1}{i}(1+i) = X\frac{(1+i)^{n+1} - (1+i)}{i}. \qquad (4.14)$$

ANNUITY FUTURE VALUES

- From the above example, find the future value of the individual’s I.R.A. if payments to the I.R.A. are made at the beginning of each year.



APPLICATION 4.2: PLANNING FOR RETIREMENT

- A 23-year-old accountant wishes to retire as a millionaire based on her retirement savings account. She intends to open and contribute to a tax-deferred retirement account sponsored by her employer each year until she retires with $1,000,000 in that account. Would she meet her retirement goal if she deposited $10,000 into that account at the end of each year until she is 65 years of age? Assume that her account will generate an annual rate of interest equal to 5% for each of the next 42 years.
- *Solution: FVn =1,352,318*

APPLICATION 4.2: PLANNING FOR RETIREMENT

- Now, suppose that she would like to retire as soon as possible with $1,000,000 in her account. Assuming that nothing else associated with her situation changes, what is the earliest age at which she can ret
- *Solution: 36.7238 ye*

DISCOUNTING AND PRESENT VALUE

- Cash flows realized at the present time have a greater value to investors than cash flows realized later, for the following reasons:
  - *Inflation*. The purchasing power of money tends to decline over time.
  - *Risk*. We never know with certainty whether we will actually realize the cash flow that we are expecting.
  - The option to either spend money now or defer spending it is likely to be worth more than being forced to defer spending the money.

(4.15)

$$PV = \frac{CF_n}{(1+k)^n},$$

where *CFn* is the cash flow to be received in year *n*, *k* is an appropriate discount rate accounting for risk, inflation, and the investor's time value associated with money, and *PV* is the present value of that cash flow.

DISCOUNTING AND PRESENT VALUE

- What would be the maximum a rational investor would be willing to pay for an investment yielding a $9,000 cash flow in six years assuming a discount rate of 15%?

THE PRESENT VALUE OF A SERIES OF CASH FLOWS

- An investor needs to evaluate a series of cash flows. She needs only to discount each separately and then sum the present values of each of the individual cash flows. Thus, the present value of a series of cash flows CF_t received in time period t can be determined by the following expression:

$$PV = \sum_{t=1}^{n} \frac{CF_t}{(1+k)^t}.$$

THE PRESENT VALUE OF A SERIES OF CASH FLOWS

- if an investment were expected to yield annual cash flows of $200 for each of the next five years, assuming a discount rate of 5%, its present value would be $865.90

$$PV = \frac{200}{(1+0.05)^1} + \frac{200}{(1+0.05)^2} + \frac{200}{(1+0.05)^3} + \frac{200}{(1+0.05)^4} + \frac{200}{(1+0.05)^5} = \$865.90.$$

$$PV_A = CF \cdot \left[\frac{1}{(1+k)^1} + \frac{1}{(1+k)^2} + \ldots + \frac{1}{(1+k)^n} \right]. \qquad (A)$$

ANNUITY PRESENT VALUES

$$PV_A = CF \cdot \left[\frac{1}{(1+k)^1} + \frac{1}{(1+k)^2} + \ldots + \frac{1}{(1+k)^n}\right]. \qquad \text{(A)}$$

multiply both sides of (A) by (1 + *k*)

$$PV_A(1+k) = CF \cdot \left[1 + \frac{1}{(1+k)^1} + \ldots + \frac{1}{(1+k)^{n-1}}\right]. \qquad \text{(B)}$$

subtract equation (A) from equation (B), to obtain

$$PV_A(1+k) - PV_A = CF\left[1 - \frac{1}{(1+k)^n}\right], \qquad \text{(C)}$$

ANNUITY PRESENT VALUES

$$PV_A = \frac{CF}{k}\left[1 - \frac{1}{(1+k)^n}\right],$$

$$PVA_{\text{due}} = \frac{CF}{k}\left[1 - \frac{1}{(1+k)^n}\right](1+k).$$

where *CF* is the level of the annual cash flow generated by the annuity (or series). The annuity formula requires that all cash flows be identical and be paid **at the end of each year.**

If cash flows were realized **at the beginning of each period**, the annuity would be referred to as an **annuity due.** Each cash flow generated by the annuity due would be received one year earlier than if cash flows were realized at the end of each year. Hence, the present value of an annuity due is determined by simply multiplying the present-value annuity formula by (1 + *k*):

APPLICATION 4.3: PLANNING FOR RETIREMENT, PART II

- Suppose that the 23-year-old accountant wishes to retire as a millionaire based on her retirement savings account. The account is open for the full 37 years, 5% discount rate.

a) its future value will be $1,016,282 . what is the present value of that million-dollar account?

b) what is the present value of the annual series $10,000 deposits that she will make to that account?

Solution:

a) $167,112.97

b) $167,112.87

APPLICATION 4.4: VALUING A BOND

- A 7% coupon bond making annual interest payments for nine years. If this bond has a $1,000 face (or par) value, and its cash flows are discounted at 6%, What can the present value of the bond's cash flows be determined?
- A 7% coupon bond will make semiannual (twice yearly) interest payments for nine years. If this bond has a $1,000 face (or par) value, and its cash flows are discounted at the stated annual rate of 6%. What can its value be determined?

AMORTIZATION

Amortization is the payment structure associated with a loan. That is, the amortization schedule of a loan is its payment schedule.

$$PV_A = \frac{CF}{k}\left[1 - \frac{1}{(1+k)^n}\right].$$

if the bank loans a sum of money equal to *PV* for *n* years at an interest rate of *i*, the amount of the annual loan repayment will be *CF*:

$$CF = [PV_A \cdot k] \div \left[1 - \frac{1}{(1+k)^n}\right].$$

AMORTIZATION

If a bank were to extend a $865,895 five year mortgage to a corporation at an interest rate of 5%, the corporation's annual payment on the mortgage would be

$$CF = [\$865{,}895 \cdot 0.05] \div \left[1 - \frac{1}{(1+0.05)^5}\right] = \$200{,}000.$$

Table 4.2 The amortization schedule of a $865,895 loan with equal annual payments for five years at 5%

Year	Principal ($)	Payment ($)	Interest ($)	Payment to principal ($)
1	865,895	200,000	43,295	156,705
2	709,189	200,000	35,459	164,541
3	544,649	200,000	27,232	172,768
4	371,881	200,000	18,594	181,406
5	190,476	200,000	9,524	190,476

APPLICATION 4.5: DETERMINING THE MORTGAGE PAYMENT

- A family has purchased a home with $30,000 down and a $300,000 mortgage. The mortgage will be amortized over 30 years with equal monthly payments. The interest rate on the mortgage will be 9% per year. Based on this data, we would like to determine the monthly mortgage payment and compile an amortization table decomposing each of the monthly payments into interest and payment toward principle.

PERPETUITY MODELS

As the value of n approaches infinity in the annuity formula, the value of the right-hand side term in the brackets,

$$\frac{1}{(1+k)^n},$$

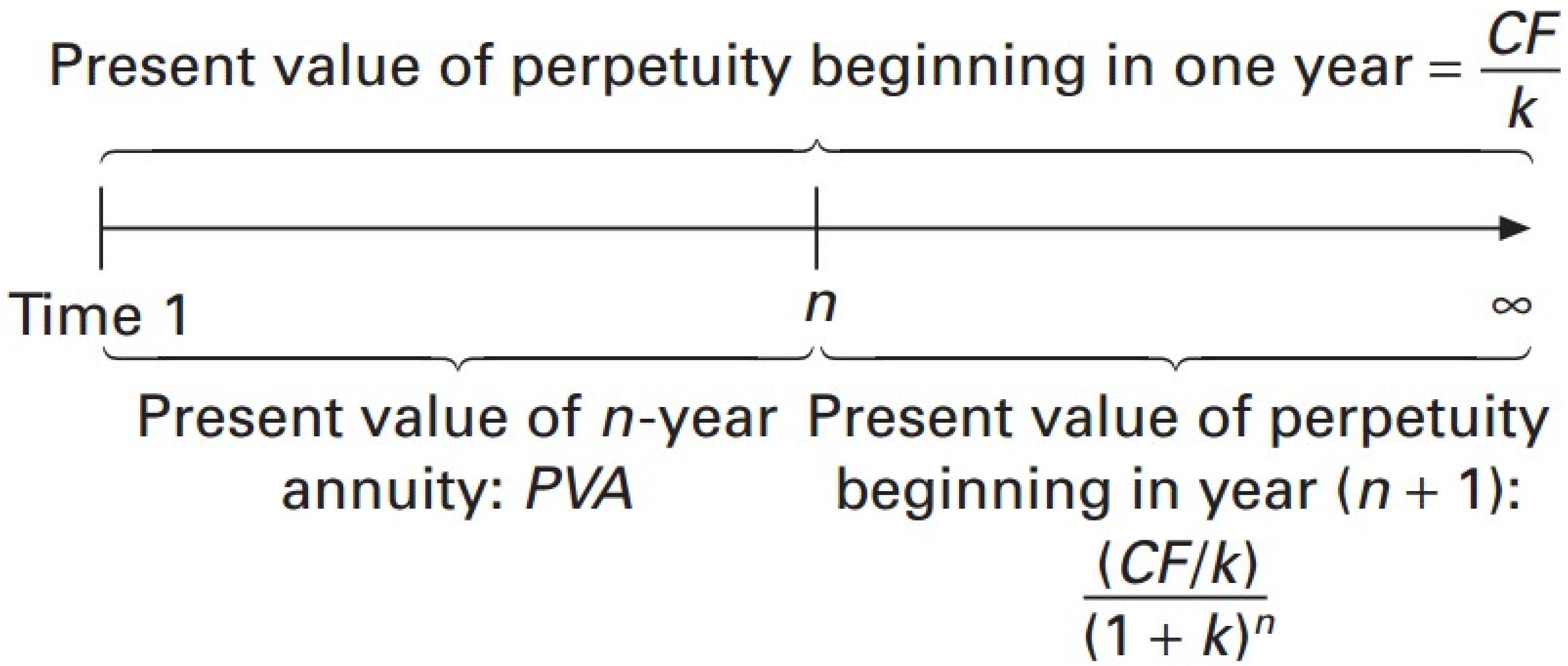
approaches zero. That is, the cash flows associated with the annuity are paid each year for a period approaching "forever." Therefore, as n approaches infinity, the value of the infinite time horizon annuity approaches

$$PV_A = \frac{CF}{k}\left[1 - \frac{1}{(1+k)^n}\right].$$

$$PV_A = \frac{CF}{k}[1-0];$$

$$PV_P = \frac{CF}{k}. \tag{4.19}$$

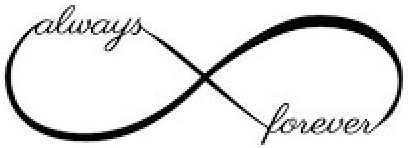
PERPETUITY MODEL



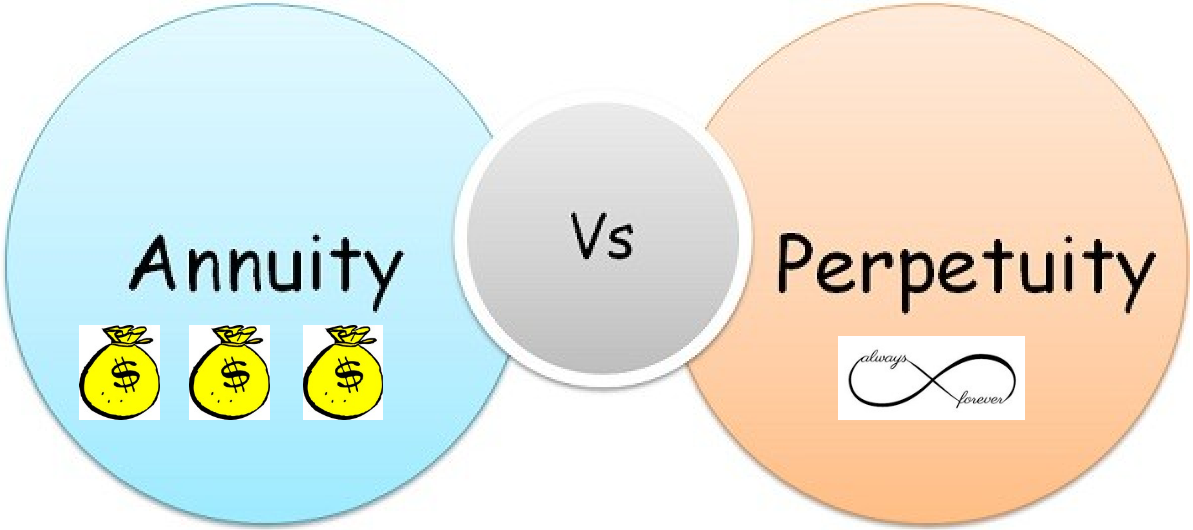
CF/k - PVA = CF/k - (CF/k*(1-1/(1+k)^n)

=(CF/k):(1+k)^n

PERPETUITY MODEL



- The present value of a perpetuity beginning in one year minus the present value of a second perpetuity beginning in year (n + 1) equals the present value of an n-year annuity.
- Thus, $PVA = CF/k \;-\; (CF/k) \div (1 + k)^n = CF/k \cdot [1 - 1/(1 + k)^n]$.



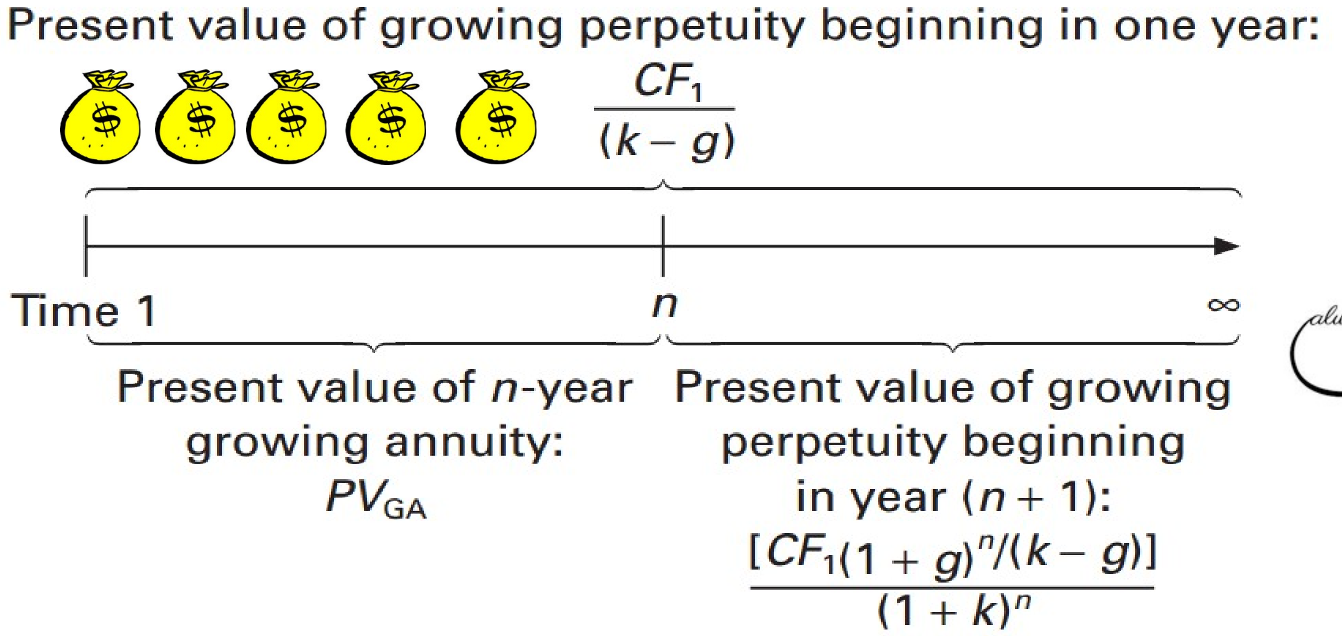
SINGLE-STAGE GROWTH MODELS

- If the cash flow associated with an investment were expected to grow at a constant annual rate of *g*, the amount of the cash flow generated by that investment in year *t* would be
  $CF_t = CF_1(1 + g)^{t-1}$,
  where *CF*1 is the cash flow generated by the investment in year one.
- *if a stock paying a dividend of $100 in year one were expected to increase its dividend payment by 10% each year thereafter, the dividend payment in the fourth year would be* $CF_4 = CF_1(1 + 0.10)^{4-1} = \$133.10$.
- *The stock's dividend in the fifth year will be* $CF_{4+1} = CF_1(1 + 0.10)^4 = \$146.41$

SINGLE-STAGE GROWTH MODELS

- If the stock had an infinite life expectancy (as most stocks might be expected to), and its dividend payments were discounted at a rate of 13%, the value of the stock would be determined by:

CF1/(k-g) = 100 /(13%



SINGLE-STAGE GROWTH MODELS

- Consider a project undertaken by a corporation whose cash flow in year one is expected to be $10,000. If cash flows were expected to grow at the inflation rate of 6% each year until year six, then terminate, the project's present value would be $48,320.35, assuming a discount rate of 11%:

$$PV_{GA} = \frac{\$10{,}000}{0.11 - 0.06}\left[1 - \frac{(1+0.06)^6}{(1+0.11)^6}\right] = \$200{,}000(1 - 0.7584) = \$48{,}320.35.$$

APPLICATION 4.6: STOCK VALUATION MODELS

- Consider a stock whose annual dividend next year is projected to be $50. This payment is expected to grow at an annual rate of 5% in subsequent years. An investor has determined that the appropriate discount rate for this stock is 10%. The current value of this stock is $1,000, determined by the growing perpetuity model:

$$PV_{gp} = \frac{\$50}{0.10 - 0.05} = \$1,000.$$

MULTIPLE-STAGE GROWTH MODELS*

- An investor has the opportunity to invest in a stock currently selling for $100 per share. The stock is expected to pay a $3 dividend next year (at the end of year 1). In each subsequent year until the seventh year, the annual dividend is expected to grow at a rate of 20%. Starting in the eighth year, the annual dividend will grow at an annual rate of 3% forever. All cash flows are to be discounted at an annual rate of 10%. Should the stock be purchased at its current price?

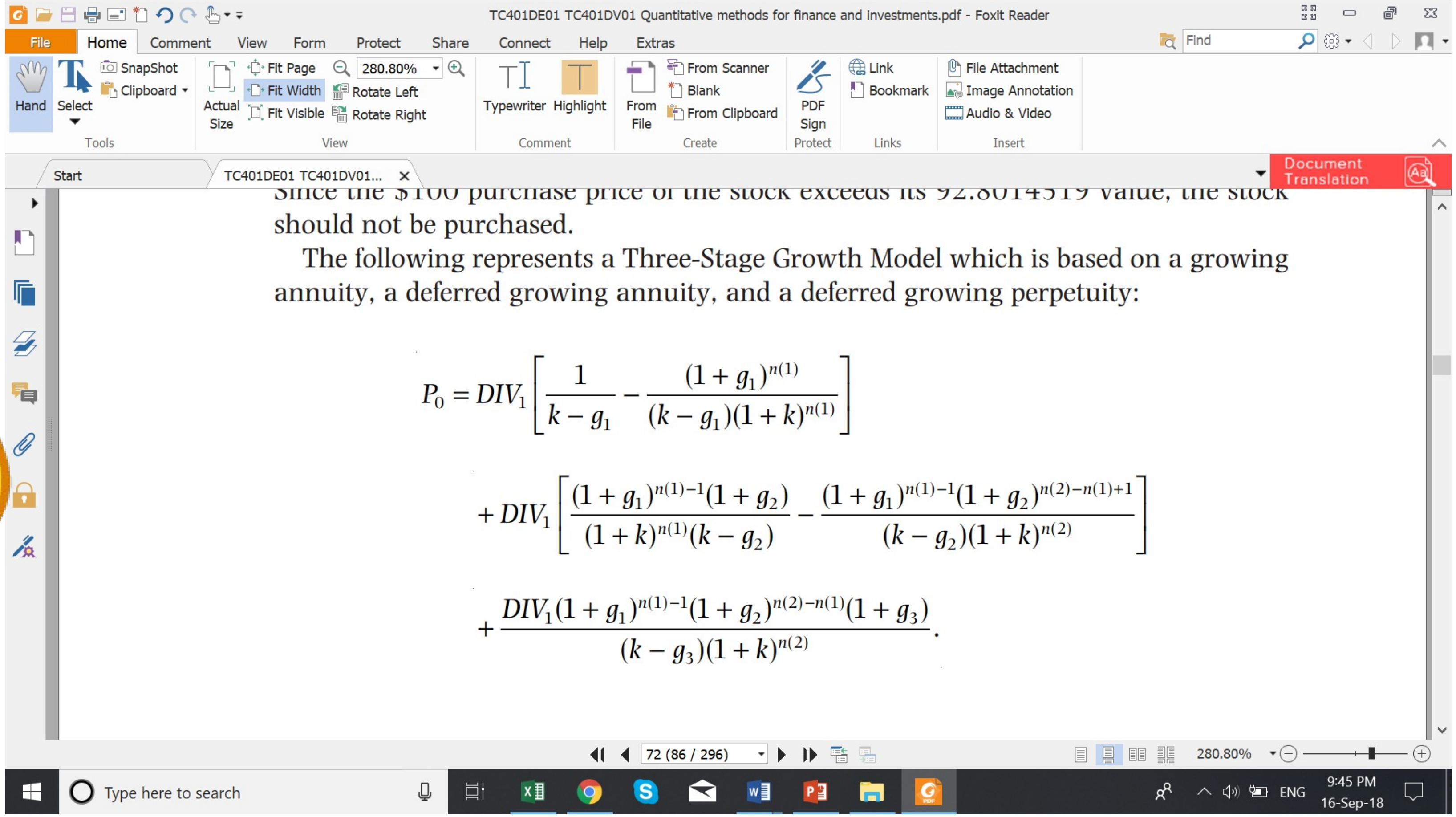
MULTIPLE-STAGE GROWTH MODELS*

The following Two-Stage Growth Model can be used to evaluate this stock:

$$P_0 = DIV_1\left[\frac{1}{k-g_1} - \frac{(1+g_1)^n}{(k-g_1)(1+k)^n}\right] + \frac{DIV_1(1+g_1)^{n-1}(1+g_2)}{(k-g_2)(1+k)^n}.$$

$$P_0 = \$3\left[\frac{1}{0.1-0.2} - \frac{(1+0.2)^7}{(0.1-0.2)(1+0.1)^7}\right] + \frac{\$3(1+0.2)^{7-1}(1+0.03)}{(0.1-0.03)(1+0.1)^7} = 92.8014519.$$

MULTIPLE-STAGE GROWTH MODELS*



Since the $100 purchase price of the stock exceeds its 92.8014519 value, the stock should not be purchased.

The following represents a Three-Stage Growth Model which is based on a growing annuity, a deferred growing annuity, and a deferred growing perpetuity:

$$P_0 = DIV_1\left[\frac{1}{k-g_1} - \frac{(1+g_1)^{n(1)}}{(k-g_1)(1+k)^{n(1)}}\right]$$

$$+ DIV_1\left[\frac{(1+g_1)^{n(1)-1}(1+g_2)}{(1+k)^{n(1)}(k-g_2)} - \frac{(1+g_1)^{n(1)-1}(1+g_2)^{n(2)-n(1)+1}}{(k-g_2)(1+k)^{n(2)}}\right]$$

$$+ \frac{DIV_1(1+g_1)^{n(1)-1}(1+g_2)^{n(2)-n(1)}(1+g_3)}{(k-g_3)(1+k)^{n(2)}}.$$